



Sharif University of Technology

Scientia Iranica

Transactions A: Civil Engineering

www.sciencedirect.com

Analysis of beam on elastic foundation using the radial point interpolation method

S.M. Binesh*Department of Civil and Environmental Engineering, Shiraz University of Technology, Shiraz, P.O. Box 71555-313, Iran*

Received 1 October 2011; revised 1 January 2012; accepted 7 January 2012

KEYWORDSMesh-free method;
Beam on elastic foundation.

Abstract The concept of beam on an elastic foundation has been extensively used by geotechnical engineers for foundation design and analysis. However, most numerical solutions for beam on an elastic foundation are obtained by mesh based methods, such as finite element or finite difference methods. Mesh based methods suffer from some deficiencies, mostly related to mesh definition. In this paper, a mesh-free method, called the radial point interpolation method, is implemented for the analysis of a beam on two parameter elastic foundation. The beam and the elastic foundation are modeled separately. The geometry of the beam is simulated by a set of nodes that are aligned on two or three parallel lines. The displacement field along the beam is constructed by radial basis functions, and the discretized system of equations is derived by substitution of the displacement field into the weak form of the governing equation. The elastic foundation is simulated by the concept of the linkage element and there is no need for nodes or elements in the traditional sense. The stiffness of the foundation has been taken into account by defining normal and tangential stiffness coefficients along the foundation layer. The displacement of each point across the foundation layer is tied to the displacement of the beam nodes. The final system of equations is derived by a combination of equations for both the beam and the elastic foundation in the global coordinate. Based on the derived equations, a computer code has been developed and the results of analysis with the mesh-free method are compared with the results of the exact solution and results obtained from finite element analysis.

© 2012 Sharif University of Technology. Production and hosting by Elsevier B.V.

Open access under [CC BY-NC-ND license](http://creativecommons.org/licenses/by-nc-nd/4.0/).

1. Introduction

There are many geotechnical engineering problems that can be idealized as beams on elastic foundations. This kind of modelling helps to understand the soil-structure interaction phenomenon and predict the contact pressure distribution and deformation within the medium. The most common theory for a beam on elastic foundation modelling is the Winkler approach [1]. However, the modelling of soil using the Winkler theory was considered inadequate in the handling of various problems. The main weakness lies in the fact that it overlooks the shear interaction between the spring elements. Hence,

some other models were proposed some decades ago [2–6], and also in recent studies [7,8]. These models have been introduced on the refinement of the Winkler approach by application of various types of interconnection between Winkler springs. One of the simplest forms of two-parameter elastic foundation models is the model with both normal and tangential stiffness along the elastic foundation [2]. The numerical solution for these two-parameter models are mainly obtained by mesh based methods, such as Finite Element (FEM) or Finite Difference Methods (FDM) [9]. Although the mesh based methods are robust, and widely used, they suffer from some deficiencies, which are mostly related to mesh definition. Hence, the author has been encouraged to implement a new class of numerical methods, which are globally coined as mesh-free or mesh-less methods, for analysis of beams on an elastic foundation.

Mesh-Free Methods (MFM) were developed by Lucy [10], who introduced Smoothed Particle Hydrodynamics (SPH) for modeling astrophysical phenomena. Libersky and Petschek [11] extended this method to solve the solid mechanics problem.

E-mail address: binesh@sutech.ac.ir.

Peer review under responsibility of Sharif University of Technology.



Production and hosting by Elsevier

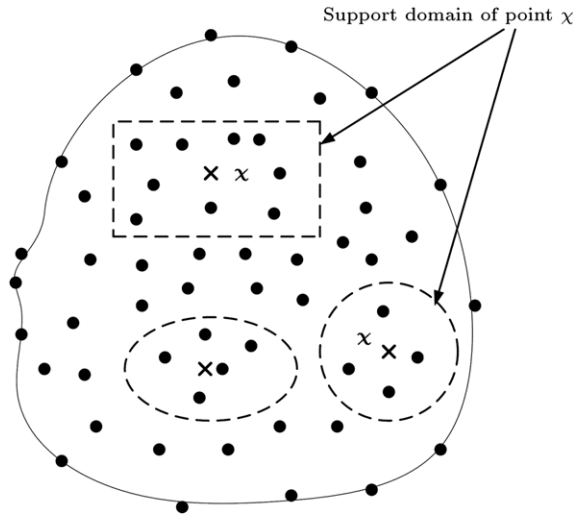


Figure 1: Support domains of different points.

Nayroles et al. [12] extended a new branch of MFMs by using a basis function and a weight function to form a local approximation based on a set of nodes. Belytschko et al. [13] improved this method by introducing the moving least square approximation and called their method “element free Galerkin”. Many other researchers proposed various MFMs, such as finite cloud [14], the reproducing kernel particle method [15], and the point interpolation method [16], etc.. Their main characteristic is that there is no need for mesh in the traditional sense.

In this paper, the enriched radial point interpolation method is implemented for the analysis of a beam on an elastic foundation with normal and tangential resistance to exerted forces. In the proposed approach, the geometry of the beam is modelled by nodes, and the displacement field is constructed by radial basis functions that are enriched by polynomial terms. In order to get rid of the extra remedies, such as the Lagrange multiplier or penalty method, the concept of a linkage element [17] has been adopted in the foundation modeling. In this approach, the two-parameter foundation is simulated by a virtual layer consisting of two sets of springs with different stiffness coefficients. The final system of equations is derived by a combination of formulations for the beam and elastic foundation. Based on the derived equations, a computer code has been developed and its validity investigated by solving some examples at the end of the paper.

2. Enriched radial point interpolation method

Polynomials have been used as basis functions in interpolation to create shape functions in many numerical methods, such as FEM. In the FEM, however, the interpolation is based on elements that have no gap and overlapping. In the point interpolation method (PIM), interpolation is based on a small set of nodes in the vicinity of a desired point, named the local support domain (Figure 1). Support domains of different points can overlap each other.

Consider a continuous function, $u(\mathbf{X})$ (i.e. displacement function). This function can be approximated in the vicinity of \mathbf{X} as follows [18]:

$$u(\mathbf{X}) = \sum_{i=1}^n p_i(\mathbf{X}) a_i = \mathbf{P}^T \mathbf{a}, \quad (1)$$

where $p_i(\mathbf{X})$ is the polynomial basis function of spatial coordinate $\mathbf{X} = [x, y]^T$, n is the number of nodes in the support domain of \mathbf{X} , and a_i is the corresponding coefficient of the basis function. In the matrix form, shown in Eq. (1),

$$\mathbf{P} = \{p_1(\mathbf{X}) \quad p_2(\mathbf{X}) \quad p_3(\mathbf{X}) \quad \cdots \quad p_n(\mathbf{X})\}^T, \quad (2)$$

$$\mathbf{a} = \{a_1 \quad a_2 \quad a_3 \quad \cdots \quad a_n\}^T, \quad (3)$$

the unknown coefficient, a_i , can be determined by enforcing $u(\mathbf{X})$ to be the nodal displacement at n nodes in the support domain. It can then be written as:

$$\mathbf{U}_S = \mathbf{P}_Q \mathbf{a}, \quad (4)$$

where \mathbf{U}_S is the vector of nodal displacements,

$$\mathbf{U}_S = \{u_1 \quad u_2 \quad u_3 \quad \cdots \quad u_n\}^T, \quad (5)$$

and \mathbf{P}_Q is the polynomial moment matrix.

$$\mathbf{P}_Q = \begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 & \cdots & p_n(\mathbf{X}_1) \\ 1 & x_2 & y_2 & x_2 y_2 & \cdots & p_n(\mathbf{X}_2) \\ 1 & x_3 & y_3 & x_3 y_3 & \cdots & p_n(\mathbf{X}_3) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & y_n & x_n y_n & \cdots & p_n(\mathbf{X}_n) \end{bmatrix}. \quad (6)$$

Assuming the existence of \mathbf{P}_Q^{-1} , a unique solution of vector \mathbf{a} can be obtained as:

$$\mathbf{a} = \mathbf{P}_Q^{-1} \mathbf{U}_S. \quad (7)$$

Substituting Eq. (7) into Eq. (1) yields:

$$u(\mathbf{X}) = \mathbf{P}^T(\mathbf{X}) \mathbf{P}_Q^{-1} \mathbf{U}_S = \sum_{i=1}^n \varphi_i u_i = \boldsymbol{\Phi}^T(\mathbf{X}) \mathbf{U}_S, \quad (8)$$

where $\boldsymbol{\Phi}(\mathbf{X})$ is the vector of PIM shape functions.

$$\boldsymbol{\Phi}^T(\mathbf{X}) = \{\varphi_1(\mathbf{X}) \quad \varphi_2(\mathbf{X}) \quad \cdots \quad \varphi_n(\mathbf{X})\}. \quad (9)$$

The shape functions constructed by PIM have the Kronecker delta function property, which allows the simple imposition of essential boundary conditions, as in conventional FEM.

The PIM is accurate and easy to use. However, an inappropriate choice of polynomial basis terms, or the bad arrangement of nodes in the support domain near \mathbf{X} will result in a singular matrix, \mathbf{P}_Q . Several strategies have been proposed to overcome this problem [18]. Using Radial Basis Functions (RBF) is one of the best solutions to guarantee the invertability of \mathbf{P}_Q . There are different functions, such as multi-quadratic, Gaussian, and Logarithmic, which can be used as RBF. In this paper, the multi-quadratic form has been used as follows:

$$R_i(\mathbf{X}) = [r_i + c^2]^q, \quad (10)$$

where r_i is the distance between the desired point (\mathbf{X}) and field node i (\mathbf{X}_i) defined as:

$$r_i = [(x - x_i)^2 + (y - y_i)^2]^{0.5}. \quad (11)$$

c and q in Eq. (10) are constants having best values that should be determined during the solution, and depend on the type of problem. For the solid mechanic problems, Liu [18] suggests 1.42 and 0.98 for c and q , respectively. The mentioned values are used in this paper.

To ensure the consistency of the radial point interpolation method (RPIM), polynomial terms are added to pure radial basis

functions. This also improves the accuracy of the results. So, the approximation of function $u(\mathbf{X})$ can be written as [18]:

$$u(\mathbf{X}) = \sum_i^n R_i(\mathbf{X})a_i + \sum_j^m p_j(\mathbf{X})b_j = \mathbf{R}^T(\mathbf{X})\mathbf{a} + \mathbf{P}^T(\mathbf{X})\mathbf{b}, \quad (12)$$

where:

$$\mathbf{R}(\mathbf{X}) = [R_1(\mathbf{X}) \ R_2(\mathbf{X}) \ R_3(\mathbf{X}) \ \cdots \ R_n(\mathbf{X})]^T, \quad (13)$$

$$\mathbf{P}(\mathbf{X}) = [p_1(\mathbf{X}) \ p_2(\mathbf{X}) \ p_3(\mathbf{X}) \ \cdots \ p_n(\mathbf{X})]^T, \quad (14)$$

$$\mathbf{a} = [a_1 \ a_2 \ a_3 \ \cdots \ a_n]^T, \quad (15)$$

$$\mathbf{b} = [b_1 \ b_2 \ b_3 \ \cdots \ b_m]^T, \quad (16)$$

where R_i and p_j are the radial and polynomial basis function, respectively, a_i and b_j are interpolation coefficients, m is the number of polynomial terms and n is the number of nodes in the support domain of \mathbf{X} . For a_i and b_j determination, $m + n$ equations are needed. The n equations can be produced by enforcing $u(\mathbf{X})$ to be the nodal displacement at n nodes in the support domain. So, we have:

$$u_k = u(x_k, y_k) = \sum_{i=1}^n R_i(x_k, y_k)a_i + \sum_{j=1}^m p_j(x_k, y_k)b_j \quad (17)$$

$k = 1, 2, \dots, n,$

or in matrix form:

$$\mathbf{U}_S = \mathbf{R}_Q \mathbf{a} + \mathbf{P}_m \mathbf{b}, \quad (18)$$

where:

$$\mathbf{R}_Q = \begin{bmatrix} R_1(r_1) & R_2(r_1) & \cdots & R_n(r_1) \\ R_1(r_2) & R_2(r_2) & \cdots & R_n(r_2) \\ \vdots & \vdots & \ddots & \vdots \\ R_1(r_n) & R_2(r_n) & \cdots & R_n(r_n) \end{bmatrix}_{n \times n}, \quad (19)$$

$$\mathbf{P}_m = \begin{bmatrix} p_1(x_1, y_1) & p_2(x_1, y_1) & \cdots & p_m(x_1, y_1) \\ p_1(x_2, y_2) & p_2(x_2, y_2) & \cdots & p_m(x_2, y_2) \\ \vdots & \vdots & \ddots & \vdots \\ p_1(x_n, y_n) & p_2(x_n, y_n) & \cdots & p_m(x_n, y_n) \end{bmatrix}_{n \times m}. \quad (20)$$

m remaining equations can be gained through:

$$\sum_{i=1}^n p_j(x_i, y_i) a_i = 0 \quad j = 1, 2, \dots, m, \quad (21)$$

which is the uniqueness condition for solutions.

In matrix form, Eq. (21) can be written as:

$$\mathbf{P}_m^T \mathbf{a} = \mathbf{0}. \quad (22)$$

Combining Eqs. (18) and (22) gives:

$$\bar{\mathbf{U}}_S = \begin{Bmatrix} \mathbf{U}_S \\ \mathbf{0} \end{Bmatrix} = \underbrace{\begin{bmatrix} \mathbf{R}_Q & \mathbf{P}_m \\ \mathbf{P}_m^T & \mathbf{0} \end{bmatrix}}_{\mathbf{G}} \begin{Bmatrix} \mathbf{a} \\ \mathbf{b} \end{Bmatrix} = \mathbf{G}\mathbf{a}_0, \quad (23)$$

where:

$$\mathbf{a}_0 = \{a_1 \ a_2 \ \cdots \ a_n \ b_1 \ b_2 \ \cdots \ b_m\}, \quad (24)$$

$$\bar{\mathbf{U}}_S = \{u_1 \ u_2 \ \cdots \ u_n \ 0 \ 0 \ \cdots \ 0\}. \quad (25)$$

From Eq. (23), we have:

$$\mathbf{a}_0 = \begin{Bmatrix} \mathbf{a} \\ \mathbf{b} \end{Bmatrix} = \mathbf{G}^{-1} \bar{\mathbf{U}}_S. \quad (26)$$

A combination of Eqs. (12) and (26) gives:

$$u(\mathbf{X}) = \{\mathbf{R}^T(\mathbf{X}) \ \mathbf{P}^T(\mathbf{X})\} \mathbf{G}^{-1} \bar{\mathbf{U}}_S = \bar{\boldsymbol{\Phi}}^T(\mathbf{X}) \bar{\mathbf{U}}_S, \quad (27)$$

where:

$$\bar{\boldsymbol{\Phi}}^T = \begin{Bmatrix} \varphi_1(\mathbf{X}) \\ \varphi_2(\mathbf{X}) \\ \vdots \\ \varphi_n(\mathbf{X}) \\ \varphi_{n+1}(\mathbf{X}) \\ \vdots \\ \varphi_{n+m}(\mathbf{X}) \end{Bmatrix}, \quad (28)$$

and the vector of shape functions is as follows:

$$\boldsymbol{\Phi}^T = \{\varphi_1(\mathbf{X}) \ \varphi_2(\mathbf{X}) \ \cdots \ \varphi_n(\mathbf{X})\}. \quad (29)$$

3. Mesh-free formulation for beam on elastic foundation

Using the beam theory, the governing differential equation for the centroidal line of the deformed beam resting on a two-parameter elastic foundation can be written as [3]:

$$EI \frac{\partial^4 y}{\partial x^4} + K_1 y - K_2 \frac{\partial^2 y}{\partial x^2} = q(x), \quad (30)$$

where E is the modulus of elasticity for the constitutive material of the beam, I is the moment of inertia for the cross section of the beam, y is the deflection of the beam at any point, x , $q(x)$ is the distributed load on the beam, and K_1 and K_2 are the first (Winkler) and the second foundation parameters, respectively.

Eq. (30) can be solved directly by strong form based methods, such as the finite difference method [19], or by weak form based methods, such as the finite element method [20]. In this paper, a weak form based method, which is also a mesh-free procedure, is implemented to solve the equation. In the present approach, the beam is considered as a two dimensional media in a plane strain condition, and it is simulated separately from the elastic foundation. Considering the problem domain entirely (i.e. beam and elastic foundation), the total potential energy functional can be expressed as:

$$\Pi = \Pi_B + \Pi_E + \Pi_F, \quad (31)$$

where Π_B and Π_E are the elastic strain energy of the beam and the foundation, respectively, and Π_F is the potential energy related to the external forces. These functionals are [21]:

$$\Pi_B = \iint_{\Omega_B} \frac{1}{2} \boldsymbol{\epsilon}_B^T \boldsymbol{\sigma}_B d\Omega_B, \quad (32)$$

$$\Pi_F = \iint_{\Omega_F} \frac{1}{2} \boldsymbol{\epsilon}_F^T \boldsymbol{\sigma}_F d\Omega_F, \quad (33)$$

$$\Pi_E = - \int_{\Gamma_B} \mathbf{U}^T \bar{\mathbf{T}} d\Gamma_B - \iint_{\Omega_B} \mathbf{U}^T \mathbf{b} d\Omega_B, \quad (34)$$

where $\boldsymbol{\epsilon}_B$ and $\boldsymbol{\sigma}_B$ are, respectively, the strain and stress tensors related to the beam,

$$\boldsymbol{\epsilon}_B^T = \{\epsilon_x \ \epsilon_y \ \gamma_{xy}\}, \quad \boldsymbol{\sigma}^T = \{\sigma_x \ \sigma_y \ \tau_{xy}\}. \quad (35)$$

$\boldsymbol{\epsilon}_F$ and $\boldsymbol{\sigma}_F$ are, respectively, the strain and stress tensors across the foundation, \mathbf{U} is the displacement vector, $\bar{\mathbf{T}}$ is the prescribed boundary traction and \mathbf{b} is the body force vector. Ω_B and Ω_F stand for the beam and foundation domains, respectively. Γ_B

is the beam boundary, on which the external tractions are imposed. It should be noted that subscripts B and F stand for the beam and foundation media, respectively.

Considering Hooke's law (i.e. $\sigma = D\epsilon$), the variational form of Eq. (31) can be written as:

$$\delta \Pi = \iint_{\Omega_B} (\delta \epsilon_B)^T D_B \epsilon_B d\Omega_B + \iint_{\Omega_F} (\delta \epsilon_F)^T D_F \epsilon_F d\Omega_F - \iint_{\Omega} (\delta U)^T b d\Omega - \int_{\Gamma} (\delta U)^T \bar{T} d\Gamma = 0, \quad (36)$$

where D_B and D_F are the elasticity matrices for the beam and foundation, respectively. The elasticity matrix for beam material in a plane strain condition can be written as [21]:

$$D_B = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix}, \quad (37)$$

where E and ν are elasticity modulus and Poisson ratio, respectively. The elasticity matrix for the foundation will be derived later in this section.

The discrete form of equations can be obtained by imposition of the strain-displacement relation and Eq. (8) into Eq. (36). For the beam material, the strain-displacement relation can be easily written as:

$$\epsilon = B U_s \quad (38)$$

where:

$$B = [\tilde{B}_{i1} \quad \tilde{B}_{i2} \quad \cdots \quad \tilde{B}_{i3} \quad \cdots \quad \tilde{B}_{in}], \quad (39)$$

and:

$$\tilde{B}_{ik} = \begin{bmatrix} \frac{\partial \varphi_{ik}}{\partial x} & 0 & \frac{\partial \varphi_{ik}}{\partial y} \\ 0 & \frac{\partial \varphi_{ik}}{\partial y} & \frac{\partial \varphi_{ik}}{\partial x} \end{bmatrix}^T, \quad (40)$$

where $\varphi_{ik} = \varphi_k(x_i)$ is the shape function of the k th node at the i th node in the support domain. However, the derivation of the stress-strain relation and, also, the strain-displacement relation in the foundation layer, requires some manipulation.

As shown in Figure 2, the two-parameter elastic foundation can be considered as a layer with two stiffness coefficients along two orthogonal directions (i.e. K_s and K_n). According to Figure 2, the relative deformation vector, δ , at point P between the top and bottom surfaces can be related to the displacements of points A and B . Hence, it can be written as:

$$\delta = \begin{bmatrix} \delta_s \\ \delta_n \end{bmatrix} = \bar{U}_A - \bar{U}_B = \begin{bmatrix} u_A - u_B \\ v_A - v_B \end{bmatrix}, \quad (41)$$

where δ_s and δ_n are, respectively, the shear and normal relative displacement at point P . \bar{U}_A and \bar{U}_B are the displacement vectors in the local coordinate $n-s$ at points A and B , respectively. However, as the location of point B is fixed, the displacement components of \bar{U}_B (i.e. u_B and v_B) are both zero. Then, Eq. (41) can be re-written as:

$$\delta = \begin{bmatrix} \delta_s \\ \delta_n \end{bmatrix} = \bar{U}_A = \begin{bmatrix} u_A \\ v_A \end{bmatrix}, \quad (42)$$

where u_A and v_A are, respectively, the displacement of point A in the s and n directions.

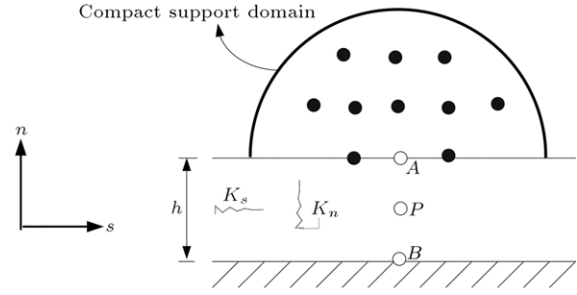


Figure 2: Foundation modeling by linkage element concept.

Considering the relation between local and global coordinates, we have:

$$\bar{U}_A = L U_A, \quad (43)$$

where L is coordinate transform matrix and U_A is the displacement vector of point A in the global coordinate.

As shown in Figure 2, the field variable (i.e. displacement) at point A can be estimated by its nodal values at the nodes located in the compact support domain of point A . According to Eqs. (8), (42) and (43), the displacement of point A can be written as:

$$\delta = B_f U_{SA}, \quad (44)$$

where U_{SA} is the displacement vector, composed of displacement at all nodes in the compact support domain of point A , and:

$$B_f = L \Phi. \quad (45)$$

Neglecting the normal strain component in the s direction, the strain vector in the local coordinate can be defined as:

$$\epsilon = \frac{1}{h} \delta, \quad (46)$$

where h is the virtual thickness assumed for the foundation layer and $\epsilon = [\gamma_{ns} \quad \epsilon_n]^T$ in which ϵ_n is the normal strain in the direction n , and γ_{ns} is the shear strain. By substituting Eq. (44) into Eq. (46), the relation between the strain and nodal displacement can be obtained as:

$$\epsilon = B_f U_{SA}, \quad (47)$$

where:

$$B_f = \left(\frac{1}{h} \right) B_f. \quad (48)$$

To evaluate the stiffness matrix related to the elastic foundation, the relation between stress and strain in this region is also needed. According to Figure 2, the relation between the stress vector and relative deformation can be written as:

$$\sigma = D_f \delta, \quad (49)$$

where the stress vector, σ , consists of the normal stress, σ_n , and the tangential stress, τ , in the foundation region.

$$\sigma = [\tau \quad \sigma_n]^T. \quad (50)$$

Matrix D_f can also be defined as:

$$D_f = \begin{bmatrix} K_s & 0 \\ 0 & K_n \end{bmatrix}. \quad (51)$$

Substituting Eq. (46) into Eq. (49) gives:

$$\sigma = D_f \epsilon, \quad (52)$$

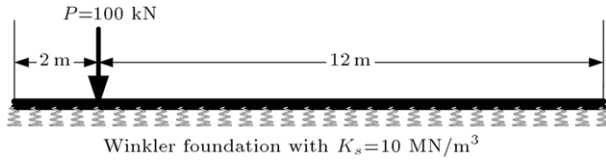


Figure 3: Beam on Winkler foundation.

where:

$$\mathbf{D}_F = \mathbf{h} \mathbf{D}_f. \quad (53)$$

At this stage, all relations between stress, strain and displacement at the beam and foundation media are known, and Eq. (36) can be rewritten as:

$$\left[\iint_{\Omega_B} \mathbf{B}_B^T \mathbf{D}_B \mathbf{B}_B d\Omega + \iint_{\Omega_F} \mathbf{B}_F^T \mathbf{D}_F \mathbf{B}_F d\Omega \right] \mathbf{U}_S = \iint_{\Omega_B} \Phi^T \mathbf{b} d\Omega_B + \int_{\Gamma_B} \Phi^T \bar{\mathbf{T}} d\Gamma_B. \quad (54)$$

Assuming constant virtual thickness for the foundation layer, Eq. (54) can be written as:

$$[\mathbf{K}_B + \mathbf{K}_f] \mathbf{U}_S = \mathbf{F}, \quad (55)$$

where:

$$\mathbf{K}_B = \iint_{\Omega_B} \mathbf{B}_B^T \mathbf{D}_B \mathbf{B}_B d\Omega_B, \quad (56)$$

$$\mathbf{K}_f = \int_{\beta} \mathbf{B}_f^T \mathbf{D}_f \mathbf{B}_f d\beta, \quad (57)$$

$$\mathbf{F} = \iint_{\Omega_B} \Phi^T \mathbf{b} d\Omega_B + \int_{\Gamma_B} \Phi^T \bar{\mathbf{T}} d\Gamma_B, \quad (58)$$

where β is the length parameter along the foundation layer.

4. Numerical study

4.1. Example 1. Beam on Winkler foundation

In this example, the well-known problem of a beam on a Winkler foundation is analyzed by the proposed mesh-free method. The problem has a closed form solution, which can be found in [9] and, hence, it is used as a benchmark for evaluation of the results of the proposed mesh-free method. As shown in Figure 3, a beam with 14 m length is considered on a Winkler foundation with the reaction modulus of 10 MN/m³. A 100 kN concentrated force is exerted at 2 m distance from the left hand side of the beam. The flexural rigidity of the beam is assumed 2604.167 kN-m², which is equivalent to a beam with 0.25 m thickness and a Young's modulus of 2×10^3 MPa at the plane strain condition. The Poisson ratio of the beam material is assumed 0.25 for numerical analysis.

To compare the results of numerical analysis with the exact solution results, a mesh-free model is constructed. As shown in Figure 4, the model consists of 58 nodes located along two parallel lines and a background mesh with 14 blocks for numerical integration. The results of analysis for both methods are depicted in Figure 5. It should be noted that the exact solution renders the deflection along the center line of the beam. Hence, the averaged values of the deflection (i.e. averaged values of deflection for the nodes located above and beneath the specified point along the center line of

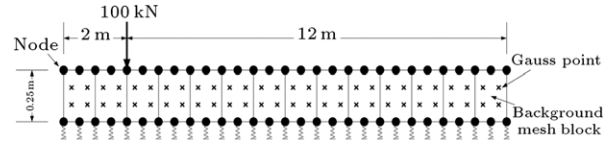


Figure 4: Mesh-free model of beam on Winkler foundation.

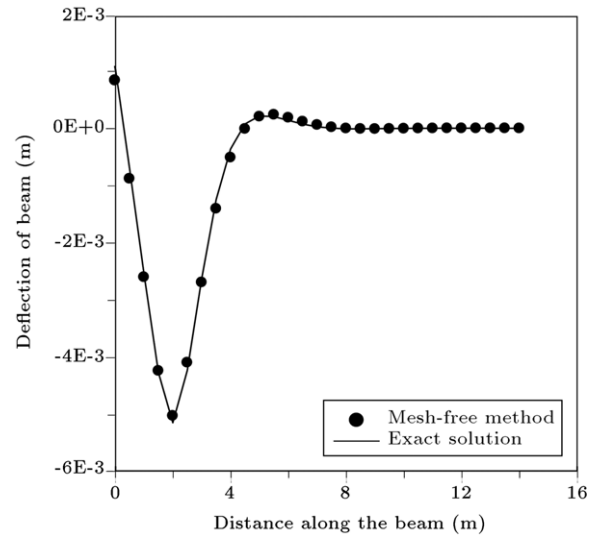


Figure 5: The deflection along the beam.

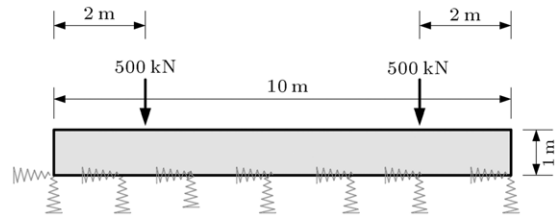


Figure 6: Typical problem of beam on two parameter elastic foundation.

the beam) obtained from numerical analysis are used for comparison. As is obvious from Figure 5, there is excellent agreement between the results of the exact solution and the proposed mesh-free method, which confirms the accuracy and efficiency of the proposed numerical solution.

4.2. Example 2. Two parameter elastic foundation

A typical problem of a beam on a two-parameter elastic foundation is investigated in the present example. As shown in Figure 6, a beam with 10 m length and 1 m thickness is assumed. The elastic modulus and Poisson ratio of the beam material are 2 GPa and 0.25, respectively. The normal and shear stiffness modulus are, respectively, 15 and 10 MN/m³. The plane strain condition is assumed. Two 500 kN concentrated loads are exerted in a symmetric manner at 2 m distance from each end.

The problem is solved by the finite element program SIGMAW [22], as well as the proposed mesh-free method. The finite element modelling consists of two models: 10 elements and 40 elements (Figure 7). It is obvious that by increasing the number of elements, the accuracy of results will also increase. Hence, this can be a measure to investigate the accuracy of

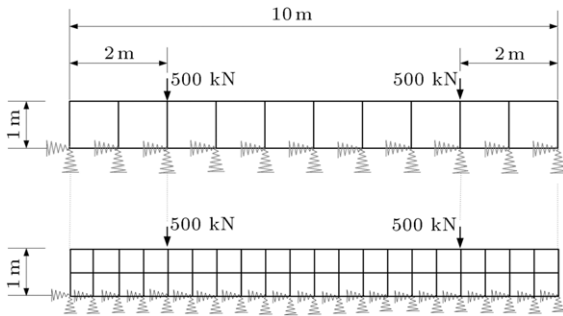


Figure 7: Finite element models of beam on two-parameter elastic foundation using 10 and 40 elements.

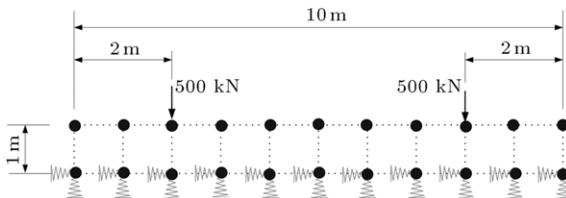


Figure 8: Mesh-free model for the beam on two-parameter elastic foundation.

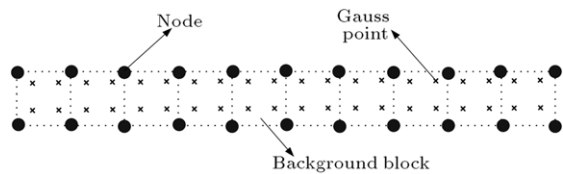


Figure 9: Background mesh blocks and Gauss point for MFM.

the proposed mesh-free method. As depicted in Figure 8, the mesh-free model is constructed by the same number of nodes as the 10 elements finite element model. The background mesh consists of 10 blocks with 4 Gauss points in each block (Figure 9). The results of analyses are shown for the upper and lower surfaces of the beam in Figures 10 and 11. Due to the symmetry of the model, only results for one half of the beam are demonstrated. As is obvious from the figures, the proposed mesh-free method offers acceptable results that are even more accurate than the results of finite element analysis with the same order of nodes. To verify this claim, quantitatively, the Mean Root Square Errors (MRSE) between different solutions are compared with each other in Table 1. In this table, the second column consists of the MRSE between the FEM with 10 elements (FEM1) and the FEM with 40 elements (FEM2), for the horizontal and vertical displacement of nodes. The third column of Table 1 shows the MRSE between MFM and FEM2 for the same components of displacement in the previous column. As obvious from values shown in Table 1, results of the mesh-free method are closer to the results of the exact solution, remembering the fact that by increasing the number of elements in FEM, the results get closer to the exact solution.

5. Conclusion

A mesh-free method is implemented for the analysis of a beam on a two parameter elastic foundation in the hope of opening a new outlook in the application of such approaches to structural analysis. The beam and elastic foundation are

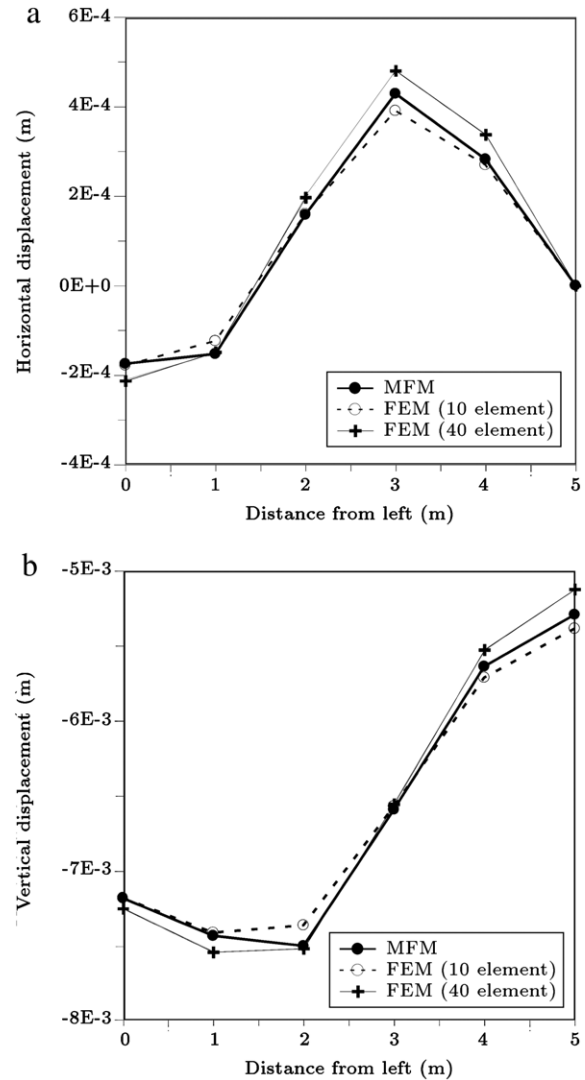


Figure 10: (a) Horizontal and (b) vertical displacement of the nodes located at the lower edge of the beam.

Table 1: Mean root square error between different solutions.

Nodal displacement	MRSE between FEM1 and FEM2 (%)	MRSE between MFM and FEM2 (%)
Horizontal displacement	19.94	14.33
Vertical displacement	2.54	1.59

simulated separately and no element is used in each part. The beam is simulated by two or three sets of nodes aligned along parallel lines. There is no connectivity between nodes, and they can be added or omitted easily. This feature gives much credit to mesh-free methods, especially in adaptivity analysis. The elastic foundation is simulated by the linkage element concept, and there is no need for nodes or elements in the traditional sense. Hence, the stresses across the foundation layer can be determined readily without any additional post processing manipulation. Besides all these benefits, the accuracy of results is also acceptable, which is even better than finite element results with the same order of nodes.

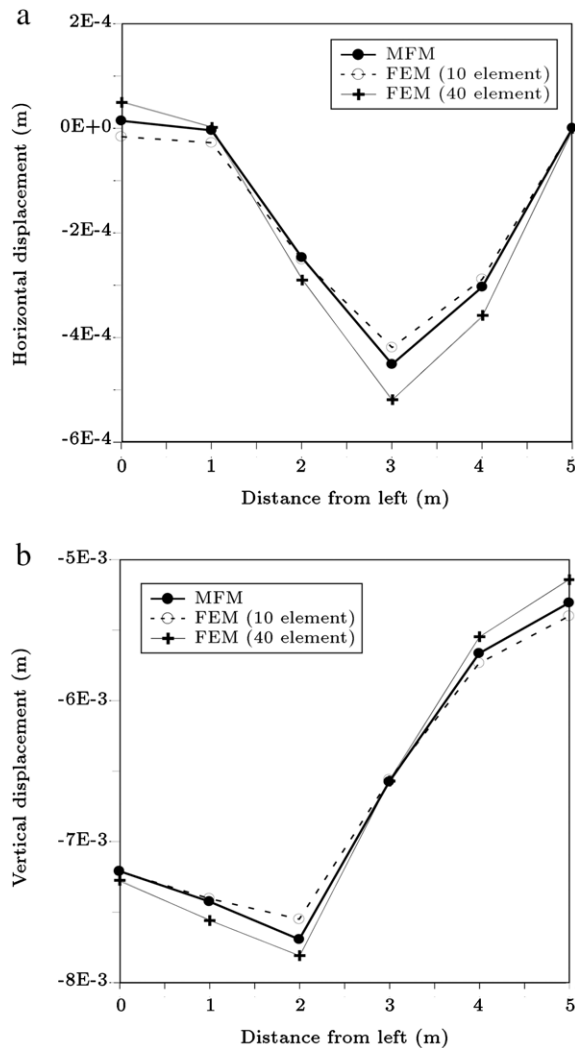


Figure 11: (a) Horizontal and (b) vertical displacement of the nodes located at the upper edge of the beam.

References

- [1] Winkler, E. *Die Lehre von der Elastizität und Festigkeit*, Domonicus, Prague (1867).
- [2] Pasternak, P.L. "On a new method of analysis of an elastic foundation by means of two foundations constants", In *Gosudarstvennoe Izdatelstvo Literaturi po Stroitelstvu Arkhitekture*, Moscow, USSR (in Russian) (1954).
- [3] Kerr, A.D. "Elastic and viscoelastic foundation models", *Journal of Applied Mechanics*, 31(3), pp. 491–498 (1964).
- [4] Zhaohua, F. and Cook, D.R. "Beam elements on two-parameter elastic foundation", *Journal of Engineering Mechanics*, 109(6), pp. 1390–1402 (1983).
- [5] Eisenberger, M. and Bielak, J. "Finite beams on infinite two-parameter elastic foundations", *Computers and Structures*, 42(4), pp. 661–664 (1992).
- [6] Pronk, A.C. "The Pasternak foundation—an attractive alternative for the Winkler foundation", *5th Int. Conf. on Concrete Pavement Design and Rehabilitation*, 1, Purdue University West Lafayette, Indiana (1993).
- [7] Morfidis, K. "Exact matrices for beams on three-parameter elastic foundation", *Computers and Structures*, 85, pp. 1243–1256 (2007).
- [8] Motaghian, S.E., Mofid, M. and Alanjari, P. "Exact solution to free vibration of beams partially supported by an elastic foundation", *Scientia Iranica*, 18(4), pp. 861–866 (2011).
- [9] Bowles, J.E., *Foundation Analysis and Design*, 5th Edn., The McGraw-Hill Companies, Inc. (1996).
- [10] Lucy, L.B. "A numerical approach to the testing of fission hypothesis", *The Astronomical Journal*, 82(12), pp. 1013–1024 (1977).
- [11] Libersky, L.D. and Petschek, A.G. "Smoothed particle hydrodynamics with strength of materials", *Proc. of the Next Free Lagrange Conf.*, pp. 248–257, (1991).
- [12] Nayroles, B., Touzot, G. and Villon, p. "Generalizing the finite element method: diffuse approximation and diffuse elements", *Computational Mechanics*, 10, pp. 307–318 (1992).
- [13] Belytschko, T., Lu, Y.Y. and Gu, L. "Element-free Galerkin methods", *International Journal for Numerical Methods in Engineering*, 37, pp. 229–256 (1994).
- [14] Duarte, C. and Oden, J.T. "An hp adaptive method using clouds", *Computer Methods in Applied Mechanics and Engineering*, 139, pp. 237–262 (1996).
- [15] Liu, W.K., Jun, S. and Zhang, Y.F. "Reproducing kernel particle methods", *International Journal for Numerical Methods in Engineering*, 20, pp. 1081–1106 (1995).
- [16] Liu, G.R. and Gu, Y.T. "A point interpolation method for two-dimensional solids", *International Journal for Numerical Methods in Engineering*, 50, pp. 937–951 (2001).
- [17] Herrmann, L.R. "Finite element analysis of contact problems", *ASCE Journal of the Engineering Mechanics Division*, 104(5), pp. 1043–1057 (1978).
- [18] Liu, G.R., *Meshfree Methods: Moving Beyond the Finite Element Method*, CRC Press, Boca (2003).
- [19] Kadivar, M.H. and Ghahramani, A. "Imbedding method for beam on Winkler foundation", *Iranian Journal of Science and Technology*, 14(1), pp. 33–46 (1990).
- [20] Zissimos, P.M. and Michael, G.P. "A finite element analysis of beams on elastic foundation including shear and axial effects", *Computers and Structures*, 27(3), pp. 323–331 (1987).
- [21] Liu, G.R. and Quek, S.S., *The Finite Element Method: A Practical Course*, 1st Edn., Elsevier Science Ltd. Jordan Hill, Oxford (2003).
- [22] SIGMAW manual "User's guide", Geo-slope International Ltd., Calgary, Alberta, Canada, (2000).

Seyyed Mohammad Binesh was born in Tehran, Iran, in 1978. He received a B.S. degree in Civil Engineering in 2000 and M.S. and Ph.D. degrees in 2002 and 2008, respectively, in Geotechnical Engineering, from Shiraz University, Iran. He is currently Assistant Professor with the Civil and Environmental Department, Shiraz University of Technology, Shiraz, Iran. His research interests include numerical methods, such as finite element and mesh-free methods, lateral loaded piles and rock mechanics.